Interdisciplinary Journal of Applied and Basic Subjects (2025), 5(5), 1-9
ISSN 2583-052X

RESEARCH PAPER

Article Identifier: https://identifier.visnav.in/1.0002/ijabs-25g-15001/

An Innovative Laplace-Based Hybrid Method for the Helmholtz PDE: Application of the Daftardar–Jafari Iteration

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Received on: 15 July 2025 *Published on: 2 October 2025*

ABSTRACT

In this paper, a new hybrid framework is proposed by combining the Laplace Transform with the Daftardar–Jafari Method for solving the Helmholtz differential equation, which is a partial differential equation widely used in modeling wave-related phenomena and is typically defined under a set of initial conditions. The proposed technique has demonstrated strong effectiveness in producing accurate and stable solutions. It has been successfully applied to several practical models arising in physics and engineering mathematics. The results confirm the efficiency and reliability of this hybrid approach in handling various types of differential equations with complex structures.

Keywords: Helmholtz equation, Daftardar-Jafari Method, Partial differential equations, Exact solution.

1. INTRODUCTION

The Helmholtz equation [1] stands as one of the most widely used partial differential equations in the mathematical modeling of physical systems due to its pivotal role in describing wave phenomena across various media. This equation arises from the analysis of vibrational and frequency-dependent behaviors such as the propagation of sound, light, and electromagnetic waves, and it finds extensive applications in scientific and engineering fields, including acoustic resonance, heat transfer in heterogeneous media, electromagnetic field analysis, and energy transmission through materials. Typically defined on bounded domains accompanied by initial and boundary conditions, the Helmholtz equation poses significant challenges in obtaining precise solutions.

Given its importance and wide range of applications, there has been a growing need to develop effective methods for deriving accurate analytical or semi-analytical solutions. In this context, several semi-analytical techniques have emerged that strike a balance between numerical simplicity and analytical precision.

Notable among these are the Homotopy Perturbation Method (HPM) [2-7], the Variational Iteration Method (VIM) [8], and other iterative approaches that construct convergent series of approximate solutions. However, many of these methods rely on the presence of small parameters or specific assumptions to facilitate the approximation process, limiting their applicability when addressing strongly nonlinear or general problems.

In contrast, the Daftardar–Jafari Method (DJM) [9-12] has gained recognition as a semi-analytical technique that does not require small parameters or restrictive initial assumptions. DJM builds a recursive series of solutions that converges rapidly to the exact solution while significantly reducing computational effort compared to traditional iterative methods. It has demonstrated remarkable effectiveness in solving a wide range of equations, including partial differential equations, integral equations, fractional differential equations, and various complex problems in physics and engineering.

In this study, we propose a hybrid approach that integrates the Laplace Transform [13] with the Daftardar–Jafari Method to solve the Helmholtz partial differential equation subject to initial conditions. The key idea lies in leveraging the Laplace Transform to convert the equation into the frequency domain, simplifying its structure and rendering it more amenable to iterative treatment. Subsequently, DJM is applied to the transformed equation to generate an approximate solution series, followed by the inverse Laplace Transform to retrieve the solution in the original domain. This fusion harnesses the robustness of the Laplace Transform and the iterative efficiency of DJM, resulting in an effective and accurate technique for tackling this fundamental equation.

The results obtained using this hybrid framework demonstrate high accuracy and notable numerical stability, underscoring its strength and suitability for applications requiring high-quality solutions, particularly in wave propagation and complex dynamical systems.

2. METHODOLOGY

Consider Helmholtz equation on a given area in the *xy*-plane of the following form:

$$\nabla^2 u + f(x,y)u = g(x,y) \quad , \quad u = u(x,y)$$

Or:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y)u = g(x,y) \tag{1}$$

Condition could be given by the following functions:

$$u(0,y) = h_1(y) \quad , \quad u_x(0,y) = h_2(y)$$

$$u(x,0) = h_3(x) \quad , \quad u_y(x,0) = h_4(x)$$

Where h_1 , h_2 , h_3 and h_4 are known functions.

Taking the Laplace transform on (1), yields:

$$s^2\, U(s,y) - su(0,y) - u_x(0,y) + U_{yy}(s,y) + L(f(x,y)u) = L\big(g(x,y)\big)$$

Then:

$$U(s,y) = \frac{1}{s}h_1(y) + \frac{1}{s^2}h_2(y) + \frac{1}{s^2}\Big(-U_{yy}(s,y) - L(f(x,y)u) + L\big(g(x,y)\big)\Big) \tag{2}$$

Taking the inverse Laplace transform of (2), yields:

$$u(x,y) = h_1(y) + xh_2(y) + L^{-1}\left(\frac{1}{s^2}\Big(-U_{yy}(s,y) - L(f(x,y)u) + L\big(g(x,y)\big)\Big)\right)$$

Where:

$$u_0(x,y) = h_1(y) + xh_2(y)$$

And the nonlinear part is:

$$N\big(u(x,y)\big) = L^{-1}\left(\frac{1}{s^2}\Big(-U_{yy}(s,y) - L\big(f(x,y)u\big) + L\big(g(x,y)\big)\Big)\right) \tag{3}$$

According to the DJM, solution of equation (3) is given in the form:

$$u = \sum_{i=0}^{\infty} u_i \tag{4}$$

By expressing the nonlinear part N in the form:

$$N\left(\sum_{i=0}^{\infty} u_i\,(x,y)\right) = L^{-1}\left(\frac{1}{s^2}\left(-\sum_{i=0}^{\infty}(U_i)_{yy}\,(s,y) - L\left(f(x,y)\sum_{i=0}^{\infty} u_i\right) + L\big(g(x,y)\big)\right)\right)$$

From this, the following results:

$$u_1 = N(u_0)$$

$$u_2 = N(u_0 + u_1) - N(u_0)$$

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$$u_m = N(u_0 + u_1 + \cdots + u_{m-1}) - N(u_0 + u_1 + \cdots + u_{m-2}) \;\;,\;\; m = 2,3,\ldots$$

3. TEST PROBLEMS

Example 1

Consider the Helmholtz equation as follows:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - u = 0 \;\;,\;\; u = u(x,y) \;\;,\;\; 0 \le x \le 1 \tag{5}$$

With the initial conditions:

$$u(0,y) = y \quad , \quad \frac{\partial u(0,y)}{\partial x} = y + \cosh(y)$$

Taking the Laplace transform on (5), yields:

$$s^2\,U(s,y) - su(0,y) - u_x(0,y) + U_{yy}(s,y) - U(s,y) = 0$$

Then:

$$U(s,y) = \frac{1}{s}y + \frac{1}{s^2}y + \frac{1}{s^2}\cosh(y) + \frac{1}{s^2}\left(-U_{yy} + U\right) \tag{6}$$

Taking the inverse Laplace transform of (6), yields:

$$u(x,y) = y + xy + x\cosh(y) + L^{-1}\left(\frac{1}{s^2}\left(-U_{yy} + U\right)\right)$$

Where:

$$u_0(x,y) = y + xy + x\cosh(y)$$

And:

$$N\big(u(x,y)\big) = L^{-1}\left(\frac{1}{s^2}\left(-U_{yy} + U\right)\right)$$

Then:

$$u_1(x,y) = L^{-1}\left(\frac{1}{s^2}\left(-(U_0)_{yy} + U_0\right)\right) = \frac{x^2}{2!}y + \frac{x^3}{3!}y$$

$$u_2(x,y) = L^{-1}\left(\frac{1}{s^2}\left(-(U_0 + U_1)_{yy} + U_0 + U_1\right)\right) - L^{-1}\left(\frac{1}{s^2}\left(-(U_0)_{yy} + U_0\right)\right) = \frac{x^4}{4!}y + \frac{x^5}{5!}y$$

Then the exact solution for (5) is:

$$u(x,y) = u_0 + u_1 + u_2 + \cdots = y\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots\right) + x\cosh(y) = y\,e^x + x\cosh(y)$$

The exact solution graph for example 1 is shown in fig.1.

Example 2

Consider the Helmholtz equation as follows:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 8u = 0 \;\;,\;\; u = u(x,y) \;\;,\;\; 0 \le x \le 1 \tag{7}$$

With the initial conditions:

$$u(0,y) = \sin(2y) \quad , \quad \frac{\partial u(0,y)}{\partial x} = 0$$

Taking the Laplace transform on (7), yields:

$$s^2\, U(s,y) - su(0,y) - u_x(0,y) + U_{yy}(s,y) + 8U(s,y) = 0$$

Then:

$$U(s,y) = \frac{1}{s}\sin(2y) - \frac{1}{s^2}\left(U_{yy} + 8U\right) \qquad (8)$$

Taking the inverse Laplace transform of (8), yields:

$$u(x,y) = \sin(2y) - L^{-1}\left(\frac{1}{s^2}\left(U_{yy} + 8U\right)\right)$$

Where:

$$u_0(x,y) = \sin(2y)$$

And:

$$N\big(u(x,y)\big) = -L^{-1}\left(\frac{1}{s^2}\left(U_{yy} + 8U\right)\right)$$

Then:

$$u_1(x,y) = -L^{-1}\left(\frac{1}{s^2}\left((U_0)_{yy} + 8U_0\right)\right) = -2x^2\sin(2y)$$

$$u_2(x,y) = -L^{-1}\left(\frac{1}{s^2}\left((U_0+U_1)_{yy} + 8(U_0+U_1)\right)\right) + L^{-1}\left(\frac{1}{s^2}\left((U_0)_{yy} + 8U_0\right)\right) = \frac{2}{3}x^4\sin(2y)$$

$$u_3(x,y) = -L^{-1}\left(\frac{1}{s^2}\left((U_0+U_1+U_2)_{yy} + 8(U_0+U_1+U_2)\right)\right) + L^{-1}\left(\frac{1}{s^2}\left((U_0+U_1)_{yy} + 8(U_0 + U_1)\right)\right) = -\frac{4}{45}x^6\sin(2y)$$

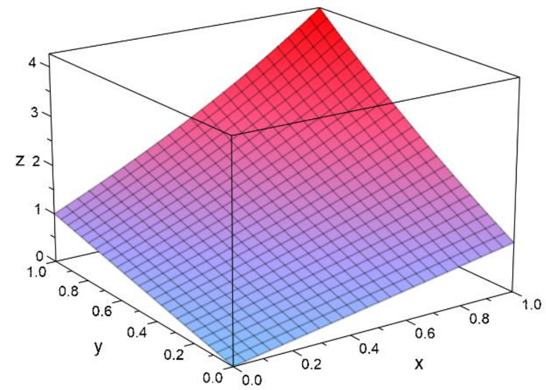

Fig 1. (axis z depicts u)

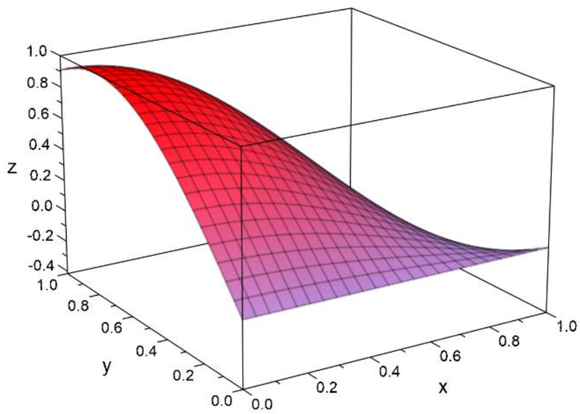

Fig 2. (axis z depicts u)

Then the exact solution for (7) is:

$$u(x,y) = u_0 + u_1 + u_2 + \cdots = \left(1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots\right)\sin(2y) = \cos(2x)\sin(2y)$$

The exact solution graph for example 2 is shown in fig.2

Example 3

Consider the Helmholtz equation as follows:

$$s^2\, U(s,y) - su(0,y) - u_x(0,y) + U_{yy}(s,y) + U(s,y) = \frac{1}{s^2}\sin(y)$$

Then:

$$U(s,y) = \frac{1}{s} + \frac{1}{s^4}\sin(y) + \frac{1}{s^2}\left(-U_{yy} - U\right) \quad (10)$$

Taking the inverse Laplace transform of (10), yields:

$$u(x,y) = 1 + \frac{x^3}{6}\sin(y) + L^{-1}\left(\frac{1}{s^2}\left(-U_{yy} - U\right)\right)$$

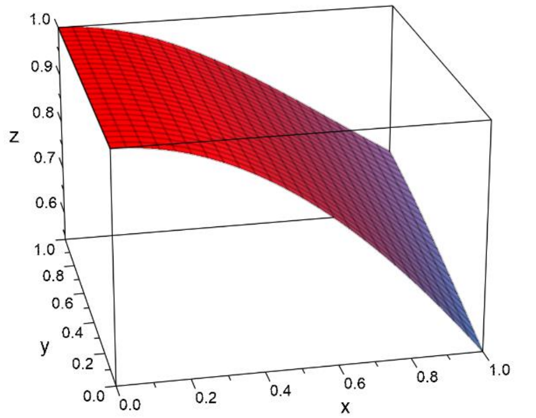

Fig 3. (axis z depicts u)

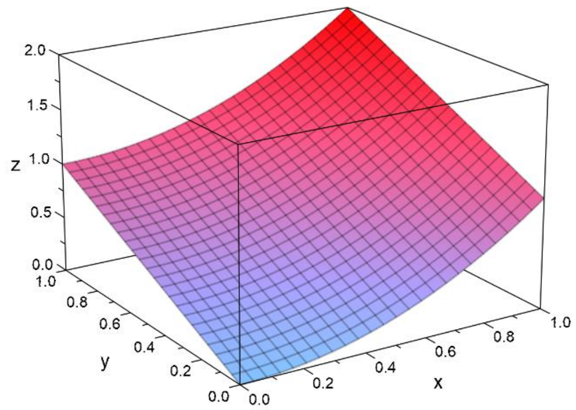

Fig 4. (axis z depicts u)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u = x\sin(y) \quad , \quad u = u(x,y) \quad , \quad 0 \le x \le 1 \quad (9)$$

With the initial conditions:

$$u(0,y) = 1 \quad , \quad \frac{\partial u(0,y)}{\partial x} = 0$$

Taking the Laplace transform on (9), yields:

Where:

$$u_0(x,y) = 1 + \frac{x^3}{6}\sin(y)$$

And:

$$N\big(u(x,y)\big) = L^{-1}\left(\frac{1}{s^2}\left(-U_{yy} - U\right)\right)$$

Then:

$$u_1(x,y) = L^{-1}\left(\frac{1}{s^2}\left(-(U_0)_{yy} - U_0\right)\right) = -\frac{x^2}{2}$$

$$u_2(x,y) = L^{-1}\left(\frac{1}{s^2}\left(-(U_0+U_1)_{yy} - (U_0+U_1)\right)\right) - L^{-1}\left(\frac{1}{s^2}\left(-(U_0)_{yy} - U_0\right)\right) = \frac{x^4}{24}$$

$$u_3(x,y) = L^{-1}\left(\frac{1}{s^2}\left(-(U_0+U_1+U_2)_{yy} - (U_0+U_1+U_2)\right)\right) - L^{-1}\left(\frac{1}{s^2}\left(-(U_0+U_1)_{yy} - (U_0 + U_1)\right)\right) = -\frac{x^6}{720}$$

Then the exact solution for (9) is:

$$u(x,y) = u_0 + u_1 + u_2 + \cdots = \frac{x^3}{6}\sin(y) + \cos(x)$$

The exact solution graph for example 3 is shown in fig.3.

Example 4

Consider the Helmholtz equation as follows:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + xu = 2 + x^3 + xy \quad , \quad u = u(x,y) \quad , \quad 0 \le x \le 1 \tag{11}$$

With the initial conditions:

$$u(0,y) = y \quad , \quad \frac{\partial u(0,y)}{\partial x} = 0$$

Taking the Laplace transform on (11), yields:

$$s^2\, U(s,y) - su(0,y) - u_x(0,y) + L\big(u_{yy} + x\,u\big) = \frac{2}{s} + \frac{6}{s^4} + \frac{1}{s^2}y$$

Then:

$$U(s,y) = \frac{1}{s}y + \frac{2}{s^3} + \frac{6}{s^6} + \frac{1}{s^4}y - \frac{1}{s^2}L\left(u_{yy} + xu\right) \tag{12}$$

Taking the inverse Laplace transform of (12), yields:

$$u(x,y) = y + x^2 + \frac{x^5}{20} + \frac{x^3}{6}y - L^{-1}\left(\frac{1}{s^2}L\left(u_{yy} + xu\right)\right)$$

Where:

$$u_0(x,y) = y + x^2 + \frac{x^5}{20} + \frac{x^3}{6}y$$

And:

$$N\big(u(x,y)\big) = -L^{-1}\left(\frac{1}{s^2}L\left(u_{yy} + xu\right)\right)$$

Then:

$$u_1(x,y) = -L^{-1}\left(\frac{1}{s^2}L\left((u_0)_{yy} + xu_0\right)\right) = -\frac{x^3}{6}y - \frac{x^5}{20} - \frac{x^8}{1120} - \frac{x^6}{180}y$$

$$u_2(x,y) = -L^{-1}\left(\frac{1}{s^2}L\left((u_0+u_1)_{yy} + x(u_0+u_1)\right)\right) + L^{-1}\left(\frac{1}{s^2}L\left((u_0)_{yy} + xu_0\right)\right) = \frac{x^8}{1120} + \frac{x^6}{180}y - \frac{x^{11}}{123200} + \cdots$$

Then the exact solution for (11) is:

$$u(x,y) = u_0 + u_1 + u_2 + \cdots = y + x^2$$

The exact solution graph for example 4 is shown in fig.4.

4. CONCLUSION AND DISCUSSION

In conclusion, it is evident that the innovative hybrid approach, combining the Laplace Transform with the Daftardar–Jafari Method, stands as a powerful and effective tool for solving the Helmholtz partial differential equation. This fusion has provided notable capabilities in addressing the equation's inherent challenges, enhancing the accuracy and stability of the solutions. The approach offers a flexible and streamlined framework that balances analytical rigor with ease of implementation, making it well-suited for applications requiring precise and complex wave modeling.

Moreover, the hybrid method was applied to solve an initial value problem represented by the Helmholtz differential equation with specified initial conditions, demonstrating its capability to handle realistic and precise mathematical problems.

This study opens broad avenues for exploring other applications that can benefit from such a hybrid methodology, especially in equations of similar or greater complexity, with potential future developments aimed at improving efficiency and solution accuracy. Given the growing demand for advanced analytical techniques, this direction remains a cornerstone for achieving deeper understanding and more reliable solutions in the field of partial differential equations.

5. ACKNOWLEDGEMENT

NA

6. CONFLICT OF INTEREST

The authors have declared that there is no conflict of interest.

7. SOURCE/S OF FUNDING

NA

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